J. Fluid Mech. (1964), *vol.* 20, *part* 2, *pp.* 315–319
Printed in Great Britain

On the non-existence of subcritical instabilities in fluid layers heated from below

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(Received 30 March 1964)

Using some recent results it is established that, for very general boundary conditions, time-independent subcritical instabilities do not exist for the non-linear thermoconvective stability problem.

1. Introduction

It has recently been rigorously established by Ukhovskii & Iudovich (1963) and Sani & Scriven (1964) that the linearized thermoconvective stability problem for a thin, bounded or unbounded, horizontal fluid layer (neglecting surface-tension effects and using the Boussinesq approximation) can be recast in a maximum (or minimum) formulation. Ukhovskii & Iudovich primarily treated the linear and non-linear stability problem for a bounded fluid layer with fixed-conducting bottom and top surfaces; Sani & Scriven treated the linear stability problem for both bounded and transversely infinite fluid layers subjected to very general boundary conditions at the top and bottom surfaces. In particular, the boundary conditions include those of a transversely infinite fluid layer subjected to a rigid bottom and free top surface with a general boundary condition of the third kind imposed on the temperature field. Here the results of these two investigations are coupled in order to establish the non-existence of a time-independent subcritical instability, i.e. an instability which can occur at a lower Rayleigh number than predicted by linear theory. Since a subcritical instability is a non-linear effect, the non-linear thermoconvective stability problem must be considered. In the present case results of linear theory coupled with an integral relation are sufficient to establish the non-existence of subcritical instabilities (see Ukhovskii & Iudovich 1963 for a particular case). This result answers the open question of the possible occurrence of subcritical hexagonal cells in a transversely infinite fluid layer with a rigid bottom and free top surface (cf. Stuart 1960*a*) and also indicates that the Stuart–Watson method for investigating finite-amplitude instabilities yields results which are in agreement with the over-all behaviour of the non-linear equations. That is, in those cases of the non-linear problem (1) and (2) which have been examined by the Stuart–Watson method (cf. Stuart 1960*b* and Watson 1960) no subcritical instabilities have been found.

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2. Basic equations

The equations characterizing a time-independent instability can be written in the form (using the Boussinesq approximations):

$$\left.\begin{aligned} \tilde{\nabla}^2\mathbf{W}+R^{\frac{1}{2}}\boldsymbol{G}.\mathbf{W}-\boldsymbol{P}.\mathbf{W} &= \boldsymbol{B}.(\mathbf{W}.\tilde{\nabla}\mathbf{W}) \\ \tilde{\nabla}.\mathbf{W} = 0, \quad \boldsymbol{P}.\mathbf{W} &= \nabla p \end{aligned}\right\} \text{ in } V, \tag{1}$$

$$\mathbf{n}.\tilde{\nabla}\mathbf{W}+\boldsymbol{A}.\mathbf{W}-R^{\frac{1}{2}}\boldsymbol{M}.\mathbf{W} = 0, \quad \text{on } \partial V, \tag{2}$$

where $\mathbf{n}$ is a unit outward-pointing normal on the surface ∂V, $\mathbf{W}$ is the hyper-vector $\mathbf{u}+\mathbf{e}T$, $\mathbf{u}$ is the velocity vector, T is the temperature excess over that which exists in the initial quiescent state, $\mathbf{e}$ is a unit abstract vector which is orthogonal to the basis vectors of physical space,

$$\left.\begin{aligned} &\tilde{\nabla}^2\mathbf{W} \equiv \nabla^2\mathbf{u}+\mathbf{e}\nabla^2 T, \quad \tilde{\nabla}.\mathbf{W} \equiv \nabla.\mathbf{u}, \\ &\boldsymbol{B} \equiv \boldsymbol{U}+N_p\,\mathbf{ee} \\ &\tilde{\nabla}\mathbf{W} \equiv \nabla\mathbf{u}+\nabla(\mathbf{e}T), \quad \boldsymbol{G} \equiv \mathbf{me}+\mathbf{em}, \end{aligned}\right\} \tag{3}$$

$\boldsymbol{U}$ is the unit dyadic of physical space, N_p is the Prandtl number, $\mathbf{m}$ is a unit vector antiparallel to the gravitational field, and $\boldsymbol{A}$ and $\boldsymbol{M}$ are linear dyadic operators which possess the following properties:

$$\left.\begin{aligned} &\oint_{\partial V}(\boldsymbol{A}\colon\mathbf{WV}-\boldsymbol{A}\colon\mathbf{VW})\,ds = 0, \quad \oint_{\partial V}\boldsymbol{A}\colon\mathbf{WW}\,ds \geqslant 0, \\ &\oint_{\partial V}(\boldsymbol{M}\colon\mathbf{WV}-\boldsymbol{M}\colon\mathbf{VW})\,ds = 0, \end{aligned}\right\} \tag{4a}$$

and the quadratic form

$$\oint_{\partial V}\boldsymbol{M}\colon\mathbf{WW}\,ds \tag{4b}$$

is either bounded with respect to the quadratic form

$$\int_V \boldsymbol{G}\colon\mathbf{WW}\,dv \tag{4c}$$

or compact with respect to the quadratic form $D(\mathbf{W};\mathbf{W})$ which is to be defined. In performing contractions on dyadic operators the convention as developed by Gibbs (see Phillips 1933) is in force. One boundary condition of the general form (2) has been used for convenience. The dyadic operators $\boldsymbol{A}$ and $\boldsymbol{M}$ will in general be functions of position on ∂V in order to satisfy conditions on the vertical boundaries which are necessary for an initial quiescent state to be realizable and to satisfy the condition $\mathbf{n}.\mathbf{u} = 0$ on ∂V which results because the physical system is closed. In the case of transversely infinite fluid layers, solutions are sought which have planform in the transverse directions (i.e. close packed cellular array) and the planform restriction can be written in the form of (2) by proper choice of $\boldsymbol{A}$ and $\boldsymbol{M}$.

3. Non-existence of subcritical instabilities

As previously mentioned it has been shown that the linearized stability problem

$$\left.\begin{aligned}\tilde{\nabla}^2\mathbf{V}+R^{\frac{1}{2}}\boldsymbol{G}.\mathbf{V} &= \boldsymbol{P}'.\mathbf{V}\\ \tilde{\nabla}.\mathbf{V}=0,\quad \boldsymbol{P}'.\mathbf{V} &= \nabla\Pi\end{aligned}\right\}\text{in } V, \tag{5}$$

with boundary condition (2) is equivalent to the extremal problem

$$\lambda_0 = \max_{\mathbf{U}\in\mathscr{W}}\frac{E(\mathbf{U};\,\mathbf{U})}{D(\mathbf{U};\,\mathbf{U})},\quad \lambda_0 \equiv (R_0^{-\frac{1}{2}}), \tag{6}$$

where λ_0 is the largest characteristic value for which a non-trivial solution exists for the linear stability problem (5) and (2) (i.e. the smallest Rayleigh number R_0). Here

$$\left.\begin{aligned}E(\mathbf{U};\,\mathbf{U}) &\equiv \int_V \boldsymbol{G}:\mathbf{UU}\,dv + \oint_{\partial V}\boldsymbol{M}:\mathbf{UU}\,ds,\\ D(\mathbf{U};\,\mathbf{U}) &\equiv \int_V \tilde{\nabla}\mathbf{U}:(\tilde{\nabla}\mathbf{U})^{\dagger}\,dv + \oint_{\partial V}\boldsymbol{A}:\mathbf{UU}\,ds\end{aligned}\right\} \tag{7}$$

(† denotes a transpose), and $\mathscr{W}$ is the Hilbert space generated by closing a set of vectors $\mathbf{U}$ which have solenoidal velocity fields, continuous second-order derivatives and satisfy the boundary condition (2) under the norm $[D(\mathbf{U};\,\mathbf{U})]^{\frac{1}{2}}$.

Now scalarly multiplying equation (1) by $\mathbf{W}$, a solution of the non-linear equations for some R,‡ and integrating over the region V, yields

$$\frac{1}{R^{\frac{1}{2}}} = \frac{E(\mathbf{W};\,\mathbf{W})}{D(\mathbf{W};\,\mathbf{W})+\int_V \boldsymbol{B}:(\mathbf{W}.\tilde{\nabla}\mathbf{W})\,\mathbf{W}\,dv}. \tag{8}$$

However, as noted by Ukhovskii & Iudovich (1963),

$$\begin{aligned}\int_V \boldsymbol{B}:(\mathbf{W}.\tilde{\nabla}\mathbf{W})\,\mathbf{W}\,dv &= \int_V [(U+N_p\mathbf{ee}):\{(\mathbf{u}+\mathbf{e}T).(\nabla\mathbf{u}+\nabla\mathbf{e}T)\}(\mathbf{u}+\mathbf{e}T)]\,dv\\ &= \int_V (\mathbf{u}+\mathbf{e}T).(\mathbf{u}.\nabla\mathbf{u}+N_p\mathbf{eu}.\nabla T)\,dv\\ &= \int_V (\mathbf{uu}:\nabla\mathbf{u}+N_p\tfrac{1}{2}\mathbf{u}.\nabla T^2)\,dv\\ &= -\frac{1}{2}\int_V(\mathbf{u}.\mathbf{u}+N_pT^2)\nabla.\mathbf{u}\,dv+\tfrac{1}{2}\oint_{\partial V}(\mathbf{u}.\mathbf{u}+N_pT^2)\mathbf{n}.\mathbf{u}\,ds = 0,\end{aligned} \tag{9}$$

because $\nabla.\mathbf{u} = 0$ in V and $\mathbf{n}.\mathbf{u} = 0$ on ∂V. Hence relation (8) becomes

$$\lambda = E(\mathbf{W};\,\mathbf{W})/D(\mathbf{W};\,\mathbf{W}),\quad \lambda \equiv R^{-\frac{1}{2}}. \tag{10}$$

By a comparison of relations (10) and (6) it follows that

$$R_0^{\frac{1}{2}} \leqslant R^{\frac{1}{2}} \tag{11}$$

and, consequently, a subcritical instability cannot occur in a thermoconvective stability problem characterized by equations (1) and (2).

‡ Also, by methods analogous to those used by Ukhovskii & Iudovich (1963) or Ladyshenskya (1963), it can be shown that for some value R a solution to equation (1) exists in $\mathscr{W}$ providing the boundary ∂V is sufficiently smooth.

4. Transversely infinite fluid layers

Next it will be shown that equality in equation (11) cannot hold for a transversely infinite fluid layer. If **W** were to yield the minimum Rayleigh number R_0, then **W** would have to satisfy the Euler equation of the extremal problem (6) whose form would be similar to equation (5) with **V** replaced by **W** and Π by Π_1. A comparison of equation (1) and the above Euler equation yields

$$\mathbf{W}.\tilde{\nabla}\mathbf{W} = \boldsymbol{Q}.\mathbf{W}, \tag{12}$$

where $\boldsymbol{Q}.\mathbf{W} = \nabla q$ and q is a scalar field. In component form equation (12) becomes

$$\left.\begin{aligned} \mathbf{u}.\nabla\mathbf{u} &= \nabla q \\ \text{and} \qquad \mathbf{u}.\nabla T &= 0. \end{aligned}\right\} \tag{13}$$

Hence, if a solution to the non-linear stability problem (1) and (2) is to exist at the minimum Rayleigh number R_0, then it would necessarily be a solution of the equation of motion of an inviscid (rotational) fluid, the transport equation $\mathbf{u}.\nabla T = 0$ and the linear stability problem with boundary conditions (2).

It can be shown that the solution of the linear stability problem for a transversely infinite horizontal fluid layer perpendicular to the x_3-axis can be represented in the form

$$\mathbf{u} = \mathbf{k}\,w(x_3)\,F(x_1,x_2) + \alpha^{-2}[Dw\,\nabla_{\mathrm{II}}F - \zeta(x_3)\,\mathbf{k}\times\nabla_{\mathrm{II}}F], \tag{14}$$

$$T = \psi(x_3)\,F(x_1,x_2). \tag{15}$$

Here $D = d/dx_3$, $\mathbf{k}.\mathbf{x} = x_3$, $\zeta(x_3)$ is the x_3-dependence of the x_3-component of vorticity, α is a real constant, $\nabla_{\mathrm{II}} \equiv \partial/\partial x_1\,\mathbf{i} + \partial/\partial x_2\,\mathbf{j}$, $\mathbf{i}.\mathbf{x} = \mathbf{x}_1$, $\mathbf{j}.\mathbf{x} = x_2$ and $F(x_1,x_2)$ is the so-called planform function which satisfies the two-dimensional scalar Helmholtz equation

$$\nabla^2_{\mathrm{II}}F = -\alpha^2 F, \tag{16}$$

with in general $F = 0$ on ∂V_2 and $\mathbf{n}.\nabla_{\mathrm{II}}F = 0$ on ∂V_3 where $\partial V_1 = \partial V_2 + \partial V_3$ is the vertical boundary of a cell and $\mathbf{n}$ is a unit outward-pointing normal on ∂V_1. Therefore, if it is required that

$$\mathbf{u}.\nabla T = 0,$$

then

$$\frac{wD\psi}{\psi Dw} = -\frac{1}{\alpha^2}\frac{\nabla_{\mathrm{II}}F.\nabla_{\mathrm{II}}F}{F^2}. \tag{17}$$

Since the left side of (17) is a function of x_3 and the right side of (17) is a function of x_1 and x_2, it follows that both must be equal to a constant β. Namely,

$$D(\ln\psi) = \beta\,D(\ln w), \tag{18}$$

$$\nabla_{\mathrm{II}}F.\nabla_{\mathrm{II}}F = -\beta\alpha^2F^2. \tag{19}$$

However, $F(x_1,x_2)$ satisfies equation (16) and the associated boundary conditions; consequently, from Green's identity it follows that

$$\iint_U \nabla_{\mathrm{II}}F.\nabla_{\mathrm{II}}F\,dA = \alpha^2\iint_U F^2\,dA, \tag{20}$$

where U is the x_1, x_2 cross-section of a cell and $dA = dx_1 dx_2$. Integrating both sides of equation (19) yields

$$\iint_U \nabla_{II} F . \nabla_{II} F \, dA = -\alpha^2 \beta \iint_U F^2 \, dA. \tag{21}$$

Comparing equations (20) and (21) yields $\beta = -1$. Substituting $\beta = -1$ into equation (18) and integrating yields

$$\psi w = b, \tag{22}$$

where $b = \text{const}$. Since $w = 0$ at $x_3 = 0, 1$, it follows from equation (22) that $b = 0$. Consequently, $\psi w = 0$ in U.

Now consider the special case where $\boldsymbol{M} \equiv \mathbf{0}$ (or $\boldsymbol{M}$ is bounded with respect to $\boldsymbol{G}$, see (4*b*) and (4*c*)). Then the condition $\psi w = 0$ in U leads to the condition $E(\mathbf{W}; \mathbf{W}) = 0$ and consequently, from the linear stability problem,

$$D(\mathbf{W}; \mathbf{W}) = 0, \tag{23}$$

if R_0 is finite. But, from equation (23), it follows that $\mathbf{W} \equiv 0$.

Consequently, a non-trivial solution of the linear stability problem for a transversely infinite horizontal fluid layer at $R = R_0$ cannot be a solution of the non-linear stability problem (1) for the special case $\boldsymbol{M} \equiv \mathbf{0}$ employed in the above discussion. In particular, this result establishes that the non-linear stability problem has only the trivial solution $\mathbf{W} \equiv 0$ for $R = R_0$ in the case of a transversely infinite horizontal fluid layer subjected to free or fixed (or combinations of both types) horizontal boundaries. Again a result which is in accord with known results of the Stuart–Watson perturbation method.

This research was sponsored by the National Science Foundation under grant GP-2337.

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